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Characterisations of compatibility, comparability and orthogonality of quantum propositions in terms of chains of filters

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Abstract. Let $p(F \circ E, \rho)$ denote the probability of a yes result of the yes-no effect obtained by putting the filter of the quantum proposition F (a projection operator) after that of the proposition E in the quantum state ρ (a statistical operator). It is pointed out that $EF = FE \Leftrightarrow \forall\rho\colon p(E \circ F, \rho) = p(F \circ E, \rho)$, $EF = E \Leftrightarrow \forall\rho\colon p(E \circ F, \rho) = p(E, \rho)$, and $EF = 0 \Leftrightarrow \forall\rho\colon p(E \circ F, \rho) = 0$. Proof is given for generalisations of these simple characterisations to arbitrary strings (products of operators) and chains (of filters). Other cases of parallelism between strings and chains and departure from it are also studied.

1. Introduction

In a previous article (Herbut 1984) an answer has been given to the old question (Birkhoff and von Neumann 1936, Mackey 1963): What is the empirical meaning of meets and joins in quantum logic? The proposed solution was given in terms of perfect filters (cf Messiah 1961, § V.13) corresponding to quantum propositions, put in series to make chains.

This study (like that in the mentioned previous article) is made from the point of view of quantum mechanics as it stands. We have a Hilbert space (state space) $\mathscr{H}$ associated with the quantum system. The projection operators E, F, G, H, etc in $\mathscr{H}$ are the quantum propositions, and the statistical operators ρ are the states. We denote the set of all ρ by $\mathscr{S}$.

It is the aim of this paper to explore a close parallelism that exists between algebraic binary relations between projection operators (quantum propositions) and probability statements on chains of perfect filters that correspond to the quantum propositions. In this way characterisations of compatibility, comparability and orthogonality of quantum propositions are obtained.

Let E and F be two quantum propositions. Putting their perfect filters in series, that of F immediately after that of E, one obtains a (composite) filter, but it is not necessarily a perfect one. If this chain ends with a detector, the number of systems that pass the composite filter can be counted, and one thus has a yes-no experiment that one writes as $F \circ E$ (read '$\circ$' as 'after'). It was shown that in the case of $H \circ G \circ \ldots \circ F \circ E$ one has for the probability of this experiment in the state ρ:

$$p(H \circ G \circ \ldots \circ F \circ E, \rho) = \mathrm{Tr}\, EF \ldots GHG \ldots FE\rho \qquad (1)$$

(see lemma 1 in Herbut 1984).

2. Strings of projection operators and chains of filters

To the idempotency $EE = E$ of a projection operator E there corresponds a parallel empirical fact in terms of filters (cf (1)):

$$\forall \rho \in \mathscr{S}: \qquad p(E \circ E, \rho) = p(E, \rho).$$

The following three theorems reveal a striking parallelism between the most important strings (products) of projection operators and the corresponding chains of filters.

Theorem 1. Two quantum propositions E and F are *compatible*, i.e.

$$EF = FE \tag{2a}$$

if and only if

$$\forall \rho \in \mathscr{S}: \qquad p(E \circ F, \rho) = p(F \circ E, \rho). \tag{2b}$$

Theorem 2. Two quantum propositions E and F are *comparable*, e.g. $E \leqslant F$, i.e.

$$EF = E \tag{3a}$$

if and only if

$$\forall \rho \in \mathscr{S}: \qquad p(E \circ F, \rho) = p(E, \rho). \tag{3b}$$

Theorem 3. The quantum propositions E and F are *orthogonal* to each other, i.e.

$$EF = 0 \tag{4a}$$

if and only if

$$\forall \rho \in \mathscr{S}: \qquad p(E \circ F, \rho) = p(0, \rho) = 0. \tag{4b}$$

We give no separate proof for theorems 1-3 because they are special cases of general results derived below.

Definition. A string (product) of two alternating projection operators E and F, e.g., $FEFEFE$, we write as $S(E, 6)$ (it begins with E reading from right to left, and its length is 6 operators). In $S(E, n)$, $S(F, n')$ we always have $n, n' \geqslant 2$. The corresponding chains (of filters) are: $F \circ E \circ F \circ E \circ F \circ E = C(E, 6)$ etc. Two strings (chains) are equal if they begin with the same projection operator (filter) and have the same length.

To derive general parallel statements on strings and chains, we need auxiliary results on two arbitrary projection operators E and F. Let us define a third projection operator G as a function of E and F such that

$$R(G^\perp) \equiv \{x: [E, F]x = 0\} \tag{5}$$

(R denotes the range, and $G^\perp \equiv 1 - G$).

Lemma 1. The subspace $R(G)$ is invariant for both E and F, and zero is the only common eigenvector of E and F in $R(G)$.

Proof. To prove the first claim it is sufficient to show that $R(G^\perp)$ is invariant. Let $b \in R(G^\perp)$. Then $[E, F]b = 0$, and $[E, F](Eb) = (EFE - FE)b = [E(EF) - FE]b =$

$[E, F]b = 0$, and symmetrically for F. The second claim follows from the fact that E and F commute when acting on a common eigenvector, and hence this vector must belong also to $R(G^\perp)$ as obvious from (5).

Lemma 2. For every $0 \neq a \in R(G)$, and every string S made up of E and F, one has $Sa = 0$ only if $Ha = 0$, where $H \equiv (E \text{ or } F)$ is the projection operator with which S begins.

Proof. We assume *ab contrario* that $\exists\, 0 \neq a \in R(G)$: $Sa = 0$ and $Ha \neq 0$. Let $S = \ldots S'$, where S' is the longest string (or only H), part of S, such that $S'a \neq 0$. Let S' end with F e.g. Then we have $F(S'a) = (S'a)$, and $E(S'a) = 0$ (because $S = \ldots ES'$) in contradiction with lemma 1.

Lemma 3. Let S and S' be two distinct strings made up of the projection operators E and F, let them both begin with the same operator, e.g. E, and let S' be longer than S. Then for each $0 \neq a \in R(G)$, we have $\|S'a\| < \|Sa\|$ unless $Ea = 0$.

Proof. Let $a \in R(G)$, $a \neq 0 \neq Ea$. Then, according to lemma 2, $Sa \neq 0$. Let S end with F, for example. Then $F(Sa) = Sa$, and this implies $E(Sa) \neq (Sa)$ due to lemma 1, resulting in $\|ESa\| < \|Sa\|$. Finally, $\|S'a\| \leq \|ESa\|$ (since $S' = \ldots ES$), and the proof is completed.

Now we are prepared to establish the mentioned generalisations.

Lemma 4. Let E and F be projection operators, and let S and S' be arbitrary distinct strings made up of of them. The quantum propositions E and F are *compatible* if and only if

$$S = S'. \tag{6a}$$

Proof. The necessity of $(6a)$ obviously follows from $(2a)$ and the idempotency of the projection operators. To prove the sufficiency of $(6a)$, we assume *ab contrario* that $[E, F] \neq 0$, i.e. that $G \neq 0$ (see (5)). Further, we first assume that both S and S' begin with the same projection operator, e.g. E, and that, for example, S' is longer than S. Taking $0 \neq a \in R(G)$, $Ea \neq 0$, lemma 3 brings us in contradiction with $(6a)$.

Secondly, we assume that, for example, S begins with E, and that S' begins with F, and that n and n' are the lengths of S and S' respectively. Let $0 \neq a \in [R(G) \cap R(E)]$. Then $Sa = \bar{S}a$, where $\bar{S}$ is the string (or F) obtained from S by dropping E from the right, and its length is $n-1$. According to lemma 1, $Fa \neq 0$, hence $\bar{S}a = S'a$ (cf $(6a)$) implies $n-1 = n'$ on account of lemma 3. Now, we take symmetrically: $0 \neq b \in [R(G) \cap R(F)]$. Then $S'b = \bar{S}'b$, etc. In this manner $(6a)$ leads to $n'-1 = n$. Both conditions on the lengths give $n-1 = n+1$, which is impossible.

Theorem 4. Let E and F be quantum propositions, and let C and C' be arbitrary distinct chains made up of their filters. Then E and F are *compatible* if and only if

$$\forall \rho \in \mathcal{S}: \qquad p(C, \rho) = p(C', \rho). \tag{6b}$$

Proof. The necessity of $(6b)$ is evident from (1) and $(2a)$. To prove its sufficiency, we rewrite (1) in the form

$$\forall \rho \in \mathscr{S}, \qquad \forall C: p(C, \rho) = \mathrm{Tr}\, S_C \rho \tag{7}$$

where S_C is the string corresponding to C in the sense of (1). Restricting oneself to pure states, it is easy to show that (6b) implies

$$S_C = S_{C'}$$

and then lemma 4 gives $[E, F] = 0$.

Putting $C \equiv E \circ F$ and $C' \equiv F \circ E$, theorem 4 gives theorem 1.

Lemma 5. Let E and F be projection operators, and let S be an arbitrary string made up of them. The quantum propositions E and F are *comparable*, e.g. $E \leqslant F$, if and only if

$$S = E. \tag{8a}$$

Proof. Evidently, (8a) follows from $E \leqslant F \Leftrightarrow EF = E$. To prove the converse implication, we assume *ab contrario* that $[E, F] \neq 0$, and we take $0 \neq a \in [R(G) \cap R(E)]$ (cf (5)). According to lemma 1, $Fa \neq a$, further S necessarily contains at least one F, hence $\|Sa\| < \|a\| = \|Ea\|$ (cf the proof of lemma 3) in contradiction with (8a). Thus, $[E, F] = 0$. Then (8a) collapses to $EF = E$.

Theorem 5. Let E and F be quantum propositions, and let C be an arbitrary chain made up of their filters. Then E and F are ***comparable***, e.g. $E \leqslant F$, if and only if

$$\forall \rho \in \mathscr{S}: \qquad p(C, \rho) = p(E, \rho). \tag{8b}$$

Proof. The necesssity of (8b) follows immediately from (1) and (3a). We obtain its sufficiency by rewriting (8b) via (7) and arriving at $S_C = E$. Then lemma 5 supplies the last step.

Substituting C by $E \circ F$ in theorem 5 we obtain theorem 2.

Lemma 6. Two quantum propositions E and F are orthogonal if and only if, taking an arbitrary string S made up of them, we have

$$S = 0. \tag{9a}$$

Proof. Relation (9a) obviously follows from orthogonality (cf (4a)). Conversely, if (9a) is valid, lemma 2 shows that $G \neq 0$ is not tenable. Owing to commutation of E and F, then, (9a) becomes $EF = 0$.

Theorem 6. Two quantum propositions E and F are *orthogonal* if and only if, taking an arbitrary chain C made up of their filters, one has

$$\forall \rho \in \mathscr{S}: \qquad p(C, \rho) = p(0, \rho) = 0. \tag{9b}$$

Proof. Orthogonality immediately implies (9b) via (1). Conversely, (9b), when rewritten in the form of (7), gives $S_C = 0$. Finally, then lemma 6 establishes the proof.

Replacing C by $E \circ F$ in theorem 6, theorem 3 is obtained.

3. More parallelisms and characterisations

Proposition 1. A quantum proposition E and a quantum state ρ stand in such a relation to each other that

$$p(E, \rho) = p(1, \rho) = 1 \tag{10a}$$

if and only if

$$E\rho = 1\rho = \rho. \tag{10b}$$

This claim was proved in appendix A of a previous article (Herbut 1969).

Proposition 2. A quantum proposition E and a quantum state ρ are such that

$$p(E, \rho) = p(0, \rho) = 0 \tag{11a}$$

if and only if

$$E\rho = 0\rho = 0. \tag{11b}$$

Proof. This is obtained by substituting E by $E^{\perp} \equiv 1 - E$ in proposition 1.

Theorem 7. One has

$$\lim_{n\to\infty} S(E, n) = \lim_{n\to\infty} S(F, n) \tag{12a}$$

and

$$\forall \rho \in \mathscr{S}: \qquad \lim_{n\to\infty} p[C(E, n), \rho] = \lim_{n\to\infty} p[C(F, n), \rho]. \tag{12b}$$

Proof. The proof of (12a) follows from the fact that the binary meet operation is symmetric:

$$E \wedge F = F \wedge E = \lim_{n\to\infty} (FEF)^n = \lim_{n\to\infty} (FE)^n. \tag{13}$$

The first expression for the meet is proved in the book of Halmos (1967). The second is obtained as follows:

$$(E \wedge F) \leqslant E \Leftrightarrow \left(\lim_{n\to\infty} (FEF)^n \right) E = \left(\lim_{n\to\infty} (FEF)^n \right),$$

$$\Rightarrow \lim_{n\to\infty} (FE)^{n+1} = \lim_{n\to\infty} (FEF)^n = E \wedge F.$$

Finally, $S(E, 2n) = (FE)^n$, $\Rightarrow \lim_{n\to\infty} S(E, n) = E \wedge F$.

As to (12b), putting $n \equiv 2k$, and utilising (1), one has

$$\text{LHS} = \lim_{k\to\infty} \operatorname{Tr}(EF)^k (FE)^k \rho = \lim_{n\to\infty} \operatorname{Tr}(EFE)^{n-1} \rho = \operatorname{Tr}(E \wedge F)\rho = \operatorname{Tr}(F \wedge E)\rho = \text{RHS}.$$

It was shown (Herbut 1984) that the LHS of (12b) is the probability in the state ρ of the quantum proposition $(E \wedge F)$ equalling the LHS of (12a).

We end this series of examples of characterisations through parallelisms with a counterexample. While for any two projection operators E and F one has

$$E = EF + EF^{\perp} \tag{14a}$$

(an identity), the corresponding relation in terms of filters is another criterion for compatibility.

Theorem 8. Two quantum propositions E and F are compatible if and only if

$$\forall \rho \in \mathscr{S}: \qquad p(E, \rho) = p(E \circ F, \rho) + p(E \circ F^{\perp}, \rho). \tag{14b}$$

Proof. Making use of (1), one rewrites (14b) as

$$\forall \rho \in \mathscr{S}: \qquad \mathrm{Tr}\ E\rho = \mathrm{Tr}\ FEF\rho + \mathrm{Tr}\ F^{\perp}EF^{\perp}\rho. \tag{15}$$

Restricting the states to pure ones, one easily establishes that (15) is equivalent to

$$E = FEF + F^{\perp}EF^{\perp}.$$

Finally, this is, in turn, obviously equivalent to $[E, F] = 0$.

4. Comments

(i) It is easy to show that in all above statements involving all states ρ, one can confine oneself to the pure states without any loss of generality. Then, owing to $\rho \equiv |\psi\rangle\langle\psi|$, (1) becomes (cf (7)):

$$p(C, |\psi\rangle) = \langle\psi|S_C|\psi\rangle.$$

Further, (10b) and (11b) take the respective forms:

$$E|\psi\rangle = 1|\psi\rangle = |\psi\rangle$$
$$E|\psi\rangle = 0|\psi\rangle = 0.$$

Thus, propositions 1 and 2 give simple physical meaning to the eigen equation of a projection operator, and indirectly (through the spectral form) to that of any Hermitian operator (observable).

(ii) Putting $C \equiv E \circ F \circ E \circ F$, and $C' = E \circ F$ in theorem 4, we see that two quantum propositions E and F are compatible if and only if the composite filter $E \circ F$ is idempotent.

(iii) Theorem 4 makes it clear that incompatibility of E and F shows up in every fixed pair of distinct chains C and C'. It is straightforward, applying lemmas 1 to 3, to find out in which states ρ this takes place.

(iv) In classical physics statement (6b) is valid for every pair of chains C and C', and for every pair of propositions E and F. In quantum mechanics (6b) is still valid for some pairs E, F (for the compatible ones). In a possible future third kind of physics a further splitting might affect also the pairs of chains. It is one of the basic aims of quantum logic to keep an eye on possible new developments of this kind.

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